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Can swell increase the number of freak waves in a wind sea?

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The effect of a swell on the statistical distribution of a directional short-wave field is investigated. Starting from Zakharov's spectral formulation, we derive a new modified nonlinear Schrödinger equation appropriate for the nonlinear evolution of a narrow-banded spectrum of short waves influenced by a swell. The swell-modified equation is solved analytically to yield an extended version of the result of Longuet-Higgins & Stewart (*J. Fluid Mech.*, vol. 8, no. 4, 1960, pp. 565–583) for the modulation of a short wave riding on a longer wave. Numerical Monte Carlo simulations of the long-term evolution of a spectrum of short waves in the presence of a monochromatic swell are employed to extract statistical distributions of freak waves among the short waves. We find evidence that a realistic short-crested wind sea can on average experience a small increase in freak wave probability because of a swell provided the swell is not orthogonal to the wind waves. For orthogonal swell and wind waves we find evidence that there is almost no significant change in the probability of freak waves in the wind sea. If the short waves are unrealistically long crested, such that the Benjamin–Feir index serves as indicator for freak waves (Gramstad & Trulsen, *J. Fluid Mech.*, vol. 582, 2007, pp. 463–472), it appears that the swell has much smaller relative influence on the probability of freak waves than in the short-crested case.

1. Introduction

It has been speculated that the interaction of a swell system and a wind-sea system can modify the probability of freak waves in comparison with either one of the wave systems alone. Recent analyses of ship accidents take such interactions into account (Toffoli *et al.* 2005) without leading to conclusions regarding the possible influence of such interaction on the probability of freak waves. In a case study of a crossing swell and wind sea, Lechuga (2006) speculated that an angle of approximately 90° between the two wave systems precluded the possibility that the swell could enhance freak wave occurrence in the wind sea.

We shall limit attention to situations in which the swell and the wind sea have different time and length scales, quite different from the crossing seas with identical periods considered by Fuhrman, Madsen & Bingham (2006), Onorato, Osborne & Serio (2006) and Shukla *et al.* (2006). We are then left with two opposite regimes of interaction that must be distinguished. For very long interaction times and lengths, there is evidence that a wind sea may modify a swell (Masson 1993) and may enhance the occurrence of freak swell waves (Tamura, Waseda & Miyazawa 2009) through

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resonant transfer of energy from the wind sea to the swell. This may happen when the wind waves and the swell have small separation in wave periods. However, in the current paper we are concerned with the opposite regime of rather short interaction times and lengths, with short waves and swell having significantly different wave periods, such that nonlinear energy transfer between short and long waves becomes unimportant. The swell can then influence the short waves through modulation of amplitude and phase, and these modulations may in turn influence the nonlinear evolution of the wind waves. The focus here is to investigate whether such a situation can lead to changes in the statistical properties of the wind waves and in particular if it makes freak waves more or less probable.

Longuet-Higgins & Stewart (1960) derived a theory for a short linear gravity wave riding on a much longer linear gravity wave. They predicted that the short wave will have locally shorter wavelength and larger amplitude close to the crests of the long wave and correspondingly longer wavelength and smaller amplitude near the troughs of the long wave. Longuet-Higgins (1987) extended this theory, allowing the linear short waves to ride on a finite-amplitude long wave. He predicted that the resulting steepening of the short waves near the crests of the long wave could be significantly enhanced.

Using a wave action approach Grimshaw (1988) considered short gravity–capillary waves riding on a steep gravity wave. Craik (1988) used a Zakharov spectral formulation for the interaction of long and short gravity waves. Both Grimshaw (1988) and Craik (1988) used linearized equations to describe the short waves. A general Hamiltonian account for the linear evolution of short waves on a long wave of a general three-dimensional form was presented by Henyey *et al.* (1988).

Extending further to allow the short waves to be weakly nonlinear, Zhang & Melville (1990) derived a nonlinear Schrödinger (NLS) equation describing weakly nonlinear short gravity waves riding on a longer finite-amplitude gravity wave. For steady short waves, they anticipated less modulation of steepness and amplitude, but more modulation of wavenumber, than compared with Longuet-Higgins (1987). Later Zhang & Melville (1992) studied the stability of weakly nonlinear short waves on finite-amplitude long waves. They found that the conventional Benjamin–Feir instability was just the first in a series of unstable regions provoked by the long wave; thus they anticipated enhanced modulational instability of short waves in the presence of long waves.

Similar results were also obtained by Naciri & Mei (1992), who considered the nonlinear evolution of short gravity waves on long rotational Gerstner waves, Naciri & Mei (1993), who considered nonlinear evolution of irrotational short waves on irrotational long waves, and Naciri & Mei (1994), who extended their consideration to two-dimensional interaction of obliquely intersecting waves. In this case the instability of the short wave due to oblique side bands was shown to be enhanced by the presence of the long wave. However, the obliqueness becomes important only when the steepness of the long wave is sufficiently large.

Recently Regev *et al.* (2008) showed that bound waves due to quadratic interaction between short waves and swell may act as an inhomogeneous disturbance in the form that is required for instability in Alber's equation (Alber 1978). By numerical integration of Alber's equation they found enhanced freak wave probability as an effect of this disturbance. The same amount of freak waves was however also found by direct simulation of the cubic NLS equation without the presence of a swell. Their investigation was limited to one horizontal dimension.

Nonlinear modulations provoked by modulational instability are known to be responsible for freak waves (Dysthe & Trulsen 1999; Akhmediev, Ankiewicz & Taki 2009). As the bandwidth of the waves becomes larger, the modulational instability will become weaker and eventually disappear (Alber 1978; Onorato *et al.* 2001; Janssen 2003; Onorato *et al.* 2004). This qualitative change of behaviour has been parameterized by the Benjamin–Feir index (BFI) defined as the ratio of steepness over bandwidth. The BFI is usually normalized such that values greater than one imply the existence of modulational instabilities that can lead to freak waves.

When considering three-dimensional waves it is found that as the crest length of waves decreases, the freak wave intensity is reduced and BFI ceases to be a useful parameter for freak wave intensity (Stansberg 1994; Onorato, Osborne & Serio 2002; Socquet-Juglard *et al.* 2005; Waseda 2006; Onorato *et al.* 2009*a*,*b*). Gramstad & Trulsen (2007) found that for waves with characteristic crest lengths shorter than about ten characteristic wavelengths the wave statistics is similar to the case of no modulational instability regardless of the value of BFI.

In this paper, our goal is to investigate whether the presence of a swell changes the extreme wave statistics for the short waves. There seems to be several different mechanisms through which a swell might influence the statistics of the short waves. First we have the swell-induced modulation of a linear monochromatic short wave discussed by Longuet-Higgins & Stewart (1960), Longuet-Higgins (1987), Henyey *et al.* (1988) and Zhang & Melville (1990). Second a weak swell may provoke the modulational instability of the short waves as suggested by Regev *et al.* (2008). Third a stronger swell may enhance the modulational instability of the short waves as found by Zhang & Melville (1992) and Naciri & Mei (1992, 1994). Finally the swell might affect the long term nonlinear evolution of the short-wave field, which is the main focus of this paper.

In order to describe a swell-modified sea, we have derived an extension of the Dysthe NLS equation suitable for describing the long-term evolution of directional short deep-water gravity waves riding on swell of arbitrary direction on finite depth. The equation is derived starting from a spectral, Zakharov-type, formulation. Details of the derivation are given in § 2.

In §§ 3 and 4 some consequences of the swell-modified equation are investigated under simplified conditions. First, in § 3 it is shown that an extended version of the result of Longuet-Higgins & Stewart (1960) can be obtained from our swell-modified equation. Then, in § 4, this result is used, under the assumption that the short waves can be considered as linear, to find approximate statistical distributions of surface and wave amplitude for a field of random short waves affected by a swell. According to this analysis, it is found that the effect of the swell on the kurtosis and probability of extreme waves is of third order in the wave steepness of the short waves, and thus a very small effect.

To further include all effects described by the swell-modified equation, numerical simulations of the full swell-modified NLS model was used in a Monte Carlo approach from which we have extracted statistics of extreme waves (Gramstad 2006). Results from the numerical simulations are presented in § 5.

In order to cover different regimes with respect to BFI and crest length, four different types of wind seas, with different BFIs and crest lengths, are considered. Furthermore, we also check the dependence of the wave statistics with respect to the relative angle of wave propagation of swell and wind sea by considering different directions of the swell relative to the short waves.

The results from the Monte Carlo simulations of the nonlinear long-term evolution reveal that the freak wave statistics of wind waves can be significantly more affected by a swell than suggested through the linear modulation mechanism of Longuet-Higgins & Stewart (1960). We find evidence that a realistically short-crested wind sea can experience an increase in freak wave probability because of a swell provided the swell is not orthogonal to the wind waves. For orthogonal swell and wind waves we find evidence that there is hardly any change in the probability of freak waves in the wind sea regardless of its crest length. On the other hand, if the short waves are unrealistically long crested such that the BFI serves as indicator for freak waves (Gramstad & Trulsen 2007), it appears that the swell has much smaller relative influence on the probability of freak waves than in the short-crested case.

2. Derivation of swell-modified equations

In this section equations describing a sea state consisting of a system of short waves propagating on top of a system of much longer waves (swell) are derived. The final result is a swell-modified NLS equation, which describes the evolution of a narrow-band wave field influenced by a system of long waves. The resulting equation was previously derived by Gramstad (2006) using a multiple-scale perturbation approach starting from the Euler equations. Here an alternative derivation based on Zakharov's spectral formulation (see e.g. Zakharov 1968; Krasitskii 1994) is presented. This method has some advantages in terms of complexity, and it provides some intermediate results which may be useful in some contexts. The present approach is in some respects similar to the approach of Craik (1988), who also considered the interaction of a long and short wave using Zakharov's spectral formulation.

Both the short waves and the swell are assumed to be described by potential theory, and the following notation is introduced for the surface elevation and velocity potential of the short waves and the swell, respectively: $\eta^s(\boldsymbol{x}, t), \phi^s(\boldsymbol{x}, z, t)$ and $\eta^l(\boldsymbol{x}, t), \phi^l(\boldsymbol{x}, z, t)$. Hence, the full surface elevation and velocity potential are $\eta = \eta^s + \eta^l$ and $\phi = \phi^s + \phi^l$, respectively.

The characteristic spatial scales for the short waves and the swell are described by the wavenumbers k_s and K for the waves and the swell respectively, where it is assumed that $K/k_s \ll 1$. Similarly, for the temporal scales we introduce the frequencies ω_s and Ω, where $\Omega/\omega_s \ll 1$ is assumed. Both the short waves and the swell are assumed to be weakly nonlinear, i.e. $k_s a_s \ll 1$, $K a_l \ll 1$, with a_s and a_l the amplitudes,

$$a_s = \sqrt{2\overline{(\eta^s - \overline{\eta^s})^2}}, \quad a_l = \sqrt{2\overline{(\eta^l - \overline{\eta^l})^2}}, \tag{2.1}$$

where the overbar represents averaging. More specifically, letting $\epsilon = k_s a_s$ be the wave steepness of the short waves, we make the scaling assumptions

$$\frac{a_s}{a_l} = O(1), \quad \frac{K}{k_s} = O(\epsilon), \quad \frac{\Omega}{\omega_s} = O(\epsilon^{1/2}), \tag{2.2}$$

which correspond to the steepness of the swell being $K a_l = O(\epsilon^2)$.

2.1. *A swell-modified Zakharov formulation*

The starting point of the derivation is the set of equations in the form

$$\hat{\eta}_t - q\hat{\psi} = 2\int E^{(3)}_{-0,1,2}\hat{\psi}_1\hat{\eta}_2\delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2} + 2\int E^{(4)}_{-0,1,2,3}\hat{\psi}_1\hat{\eta}_2\hat{\eta}_3\delta_{0-1-2-3}\,\mathrm{d}\boldsymbol{k}_{1,2,3}, \qquad (2.3a)$$

$$\hat{\psi}_t + g\hat{\eta} = -\int E^{(3)}_{1,2,-0}\hat{\psi}_1\hat{\psi}_2\delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2} - 2\int E^{(4)}_{1,2,3,-0}\hat{\psi}_1\hat{\psi}_2\hat{\eta}_3\delta_{0-1-2-3}\,\mathrm{d}\boldsymbol{k}_{1,2,3}. \qquad (2.3b)$$

These equations are given in Krasitskii (1994) and describe the nonlinear evolution of the variables $\hat{\eta}(\boldsymbol{k},t)$ and $\hat{\psi}(\boldsymbol{k},t)$; the spatial Fourier transforms of the free surface and the velocity potential at the free surface, respectively. Here, $k = |\boldsymbol{k}|$ and $q(\boldsymbol{k}) = k\tanh(kh)$. Furthermore, $\hat{\eta}_t$ and $\hat{\psi}_t$ denote the partial derivative with respect to time of $\hat{\eta}$ and $\hat{\psi}$ respectively. We have adopted a compact subscript notation, e.g. $E^{(3)}_{0,1,2} = E^{(3)}(\boldsymbol{k}_0,\boldsymbol{k}_1,\boldsymbol{k}_2)$, $\delta_{0-1-2} = \delta(\boldsymbol{k}_0 - \boldsymbol{k}_1 - \boldsymbol{k}_2)$ and $\mathrm{d}\boldsymbol{k}_{1,2} = \mathrm{d}\boldsymbol{k}_1\mathrm{d}\boldsymbol{k}_2$. However, the complete notation is also used whenever it is convenient. The kernel functions $E^{(3)}_{0,1,2}$ and $E^{(4)}_{0,1,2,3}$ can be found in Krasitskii (1994).

The surface elevation and velocity potential of the waves and the swell could be expressed as the following Fourier transforms:

$$\eta^{s,l}(\boldsymbol{x},t) = \frac{1}{2\pi}\int \hat{\eta}^{s,l}(\boldsymbol{k},t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k}, \quad \phi^{s,l}(\boldsymbol{x},z,t) = \frac{1}{2\pi}\int \hat{\phi}^{s,l}(\boldsymbol{k},t)\frac{\cosh[k(z+h)]}{\sinh[kh]}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k}. \qquad (2.4)$$

Since the surface elevation and velocity potential are real functions it is clear that $\hat{\eta}^{s,l}(\boldsymbol{k},t) = \hat{\eta}^{s,l*}(-\boldsymbol{k},t)$ and $\hat{\phi}^{s,l}(\boldsymbol{k},t) = \hat{\phi}^{s,l*}(-\boldsymbol{k},t)$. Further, we define the surface potentials

$$\psi^{l_0} = \phi^l(z=\eta^l), \quad \psi^s = \phi^s(z=\eta^s+\eta^l), \quad \psi^l = \phi^l(z=\eta^s+\eta^l). \qquad (2.5)$$

We note that the full surface potential is $\psi = \phi(z=\eta) = \psi^l + \psi^s$.

If we make use of the separation between the short waves and the swell, (2.3*a*) and (2.3*b*) can be written as

$$\hat{\eta}^s_t - q\hat{\psi}^s + \hat{\eta}^l_t - q\hat{\psi}^l = 2\int E^{(3)}_{-0,1,2}(\hat{\psi}^s_1 + \hat{\psi}^l_1)(\hat{\eta}^s_2 + \hat{\eta}^l_2)\delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2}$$
$$+ 2\int E^{(4)}_{-0,1,2,3}(\hat{\psi}^s_1 + \hat{\psi}^l_1)(\hat{\eta}^s_2 + \hat{\eta}^l_2)(\hat{\eta}^s_3 + \hat{\eta}^l_3)\delta_{0-1-2-3}\,\mathrm{d}\boldsymbol{k}_{1,2,3}, \qquad (2.6a)$$

$$\hat{\psi}^s_t + g\hat{\eta}^s + \hat{\psi}^l_t + g\hat{\eta}^l = -\int E^{(3)}_{1,2,-0}(\hat{\psi}^s_1 + \hat{\psi}^l_1)(\hat{\psi}^s_2 + \hat{\psi}^l_2)\delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2}$$
$$- 2\int E^{(4)}_{1,2,3,-0}(\hat{\psi}^s_1 + \hat{\psi}^l_1)(\hat{\psi}^s_2 + \hat{\psi}^l_2)(\hat{\eta}^s_3 + \hat{\eta}^l_3)\delta_{0-1-2-3}\,\mathrm{d}\boldsymbol{k}_{1,2,3}. \qquad (2.6b)$$

We assume that the swell exists independent of the short waves; i.e. $\hat{\eta}^l$ and $\hat{\psi}^{l_0}$ satisfy (2.3*a*) and (2.3*b*) separately. In order to make use of this assumption, we note that

Taylor expansion of $\psi^l = \phi^l(z = \eta^s + \eta^l)$ gives

$$\psi^l = \phi^l(z = \eta^l) + \eta^s \left.\frac{\partial \phi^l}{\partial z}\right|_{z=0} + \frac{\eta^s}{2}\left(\eta^s + 2\eta^l\right) \left.\frac{\partial^2 \phi^l}{\partial z^2}\right|_{z=0} + \cdots. \tag{2.7}$$

By taking the Fourier transform of (2.7), we obtain

$$\hat{\psi}^l = \hat{\psi}^{l_0} + \frac{1}{2\pi}\int k_1 \hat{\phi}^l_1 \hat{\eta}^s_2 \delta_{0-1-2}\, \mathrm{d}\boldsymbol{k}_{1,2} + \frac{1}{(2\pi)^2}\int \frac{k_1^2}{2} \coth\,[k_1 h] \hat{\phi}^l_1 \hat{\eta}^s_2 (\hat{\eta}^s_3 + 2\hat{\eta}^l_3) \delta_{0-1-2-3}\, \mathrm{d}\boldsymbol{k}_{1,2,3} + \cdots. \tag{2.8}$$

Similarly, one can also show that

$$\hat{\psi}^{l_0} = \coth\,[kh]\hat{\phi}^l_0 + \frac{1}{2\pi}\int k_1 \hat{\phi}^l_1 \hat{\eta}^l_2 \delta_{0-1-2}\, \mathrm{d}\boldsymbol{k}_{1,2} + \frac{1}{(2\pi)^2}\int \frac{k_1^2}{2} \coth\,[k_1 h] \hat{\phi}^l_1 \hat{\eta}^l_2 \hat{\eta}^l_3 \delta_{0-1-2-3}\, \mathrm{d}\boldsymbol{k}_{1,2,3} + \cdots. \tag{2.9}$$

We now insert (2.8) into (2.6*a*) and (2.6*b*) and make use of the assumption that the swell exists independent of the short waves. We also make use of the introductory scaling assumptions and only account for terms up to third order in the wave steepness, ϵ, of the short waves. This gives

$$\hat{\eta}^s_t - q\hat{\psi}^s - \frac{q}{2\pi}\int k_1 \hat{\phi}^l_1 \hat{\eta}^s_2 \delta_{0-1-2}\, \mathrm{d}\boldsymbol{k}_{1,2} = 2\int E^{(3)}_{-0,1,2}\left[\hat{\psi}^s_1 \hat{\eta}^s_2 + \hat{\psi}^s_1 \hat{\eta}^l_2 + \hat{\psi}^{l_0}_1 \hat{\eta}^s_2\right]\delta_{0-1-2}\, \mathrm{d}\boldsymbol{k}_{1,2} + 2\int E^{(4)}_{-0,1,2,3} \hat{\psi}^s_1 \hat{\eta}^s_2 \hat{\eta}^s_3 \delta_{0-1-2-3}\, \mathrm{d}\boldsymbol{k}_{1,2,3}, \tag{2.10a}$$

$$\hat{\psi}^s_t + g\hat{\eta}^s + \frac{1}{2\pi}\int k_1 \left[\hat{\phi}^l_1 \hat{\eta}^s_2\right]_t \delta_{0-1-2}\, \mathrm{d}\boldsymbol{k}_{1,2} = -\int E^{(3)}_{1,2,-0}\left[\hat{\psi}^s_1 \hat{\psi}^s_2 + \hat{\psi}^s_1 \hat{\psi}^{l_0}_2 + \hat{\psi}^{l_0}_1 \hat{\psi}^s_2\right]\delta_{0-1-2}\, \mathrm{d}\boldsymbol{k}_{1,2} - 2\int E^{(4)}_{1,2,3,-0} \hat{\psi}^s_1 \hat{\psi}^s_2 \hat{\eta}^s_3 \delta_{0-1-2-3}\, \mathrm{d}\boldsymbol{k}_{1,2,3}. \tag{2.10b}$$

With use of (2.9) this may be rewritten in the form

$$\hat{\eta}^s_t - q\hat{\psi}^s = 2\int E^{(3)}_{-0,1,2} \hat{\psi}^s_1 \hat{\eta}^s_2 \delta_{0-1-2}\, \mathrm{d}\boldsymbol{k}_{1,2} + 2\int E^{(4)}_{-0,1,2,3} \hat{\psi}^s_1 \hat{\eta}^s_2 \hat{\eta}^s_3 \delta_{0-1-2-3}\, \mathrm{d}\boldsymbol{k}_{1,2,3} + 2\int \tilde{E}^{(3)}_{-0,1,2} \coth\,[k_1 h] \hat{\phi}^l_1 \hat{\eta}^s_2 \delta_{0-1-2}\, \mathrm{d}\boldsymbol{k}_{1,2} + 2\int E^{(3)}_{-0,1,2} \hat{\psi}^s_1 \hat{\eta}^l_2 \delta_{0-1-2}\, \mathrm{d}\boldsymbol{k}_{1,2}, \tag{2.11a}$$

$$\hat{\psi}^s_t + g\hat{\eta}^s = -\int E^{(3)}_{1,2,-0} \hat{\psi}^s_1 \hat{\psi}^s_2 \delta_{0-1-2}\, \mathrm{d}\boldsymbol{k}_{1,2} - 2\int E^{(4)}_{1,2,3,-0} \hat{\psi}^s_1 \hat{\psi}^s_2 \hat{\eta}^s_3 \delta_{0-1-2-3}\, \mathrm{d}\boldsymbol{k}_{1,2,3} - 2\int \tilde{E}^{(3)}_{1,2,-0} \coth\,[k_2 h] \hat{\psi}^s_1 \hat{\phi}^l_2 \delta_{0-1-2}\, \mathrm{d}\boldsymbol{k}_{1,2} + \frac{1}{2\pi}\int \omega_1^2 \hat{\eta}^l_1 \hat{\eta}^s_2 \delta_{0-1-2}\, \mathrm{d}\boldsymbol{k}_{1,2}, \tag{2.11b}$$

where $\tilde{E}^{(3)}_{0,1,2} = E^{(3)}_{0,1,2} + q_0 q_1/4\pi = -\boldsymbol{k}_0 \cdot \boldsymbol{k}_1/4\pi$.

We now introduce the complex amplitude function $a(\boldsymbol{k}, t)$,

$$\hat{\eta}^s(\boldsymbol{k}, t) = \mathscr{M}(\boldsymbol{k})\left[a(\boldsymbol{k}, t) + a^*(-\boldsymbol{k}, t)\right], \quad \hat{\psi}^s(\boldsymbol{k}, t) = -\mathrm{i}\mathscr{N}(\boldsymbol{k})\left[a(\boldsymbol{k}, t) - a^*(-\boldsymbol{k}, t)\right], \tag{2.12}$$

or alternatively $a(\boldsymbol{k},t)=\mathscr{N}(\boldsymbol{k})\hat{\eta}^s(\boldsymbol{k},t)+\mathrm{i}\mathscr{M}(\boldsymbol{k})\hat{\psi}^s(\boldsymbol{k},t)$, where

$$\mathscr{M}(\boldsymbol{k})=\left[\frac{\omega(\boldsymbol{k})}{2g}\right]^{1/2},\quad \mathscr{N}(\boldsymbol{k})=\left[\frac{g}{2\omega(\boldsymbol{k})}\right]^{1/2},\quad \omega(\boldsymbol{k})=\sqrt{gk\tanh(kh)}. \tag{2.13}$$

Now, by combining (2.11*a*) and (2.11*b*), we obtain the following equation for $a(\boldsymbol{k},t)$:

$$\begin{aligned}
\mathrm{i}\frac{\partial a_0}{\partial t}-\omega_0 a_0 &= \int U^{(1)}_{0,1,2}a_1a_2\delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2}+2\int U^{(1)}_{2,1,0}a_1^*a_2\delta_{0+1-2}\,\mathrm{d}\boldsymbol{k}_{1,2}\\
&+\int U^{(3)}_{0,1,2}a_1^*a_2^*\delta_{0+1+2}\,\mathrm{d}\boldsymbol{k}_{1,2}+\int V^{(1)}_{0,1,2,3}a_1a_2a_3\delta_{0-1-2-3}\,\mathrm{d}\boldsymbol{k}_{1,2,3}\\
&+\int V^{(2)}_{0,1,2,3}a_1^*a_2a_3\delta_{0+1-2-3}\,\mathrm{d}\boldsymbol{k}_{1,2,3}+3\int V^{(2)}_{3,2,1,0}a_1^*a_2^*a_3\delta_{0+1+2-3}\,\mathrm{d}\boldsymbol{k}_{1,2,3}\\
&+\int V^{(4)}_{0,1,2,3}a_1^*a_2^*a_3^*\delta_{0+1+2+3}\,\mathrm{d}\boldsymbol{k}_{1,2,3}+4\mathrm{i}\int \mathscr{M}_1 S^{(1)}_{0,1,2}\coth[k_1h]\hat{\phi}_1^l a_2\delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2}\\
&+4\int \mathscr{N}_1 S^{(2)}_{0,1,2}\hat{\eta}_1^l a_2\delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2}+4\mathrm{i}\int \mathscr{M}_1 S^{(1)}_{0,1,2}\coth[k_1h]\hat{\phi}_1^l a_2^*\delta_{0-1+2}\,\mathrm{d}\boldsymbol{k}_{1,2}\\
&+4\int \mathscr{N}_1 S^{(3)}_{0,1,2}\hat{\eta}_1^l a_2^*\delta_{0-1+2}\,\mathrm{d}\boldsymbol{k}_{1,2}.
\end{aligned} \tag{2.14}$$

The first seven integrals on the right-hand side are the same as in the 'wave-only' problem, and the functions $U^{(1)}$, $U^{(3)}$, $V^{(1)}$, $V^{(2)}$ and $V^{(4)}$ can be found in Krasitskii (1994). The last four integrals are due to the wave–swell interaction, and the functions $S^{(1)}$, $S^{(2)}$ and $S^{(3)}$ are found to be

$$\begin{aligned}
&S^{(1)}_{0,1,2}=\tilde{U}_{2,1,-0}-\tilde{U}_{-0,1,2},\quad S^{(2)}_{0,1,2}=-U_{-0,2,1}-\frac{1}{4\pi}\mathscr{M}_0\mathscr{M}_1\mathscr{M}_2\omega_1^2,\\
&S^{(3)}_{0,1,2}=U_{0,2,1}-\frac{1}{4\pi}\mathscr{M}_0\mathscr{M}_1\mathscr{M}_2\omega_1^2,
\end{aligned}$$

where $\tilde{U}_{0,1,2}=-\mathscr{N}_0\mathscr{N}_1\mathscr{M}_2\tilde{E}^{(3)}_{0,1,2}$.

Now, in light of Krasitskii (1994), we attempt to remove terms in (2.14) that correspond to interactions that are far from resonance, by employing a procedure similar to the canonical transformation described in Krasitskii (1994). The result will be an equation in the 'free-wave' amplitude function $b(\boldsymbol{k},t)$, which is assumed to be related to $a(\boldsymbol{k},t)$ through the following relation:

$$\begin{aligned}
a_0 &= b_0+\int C^{(1)}_{0,1,2}b_1b_2\delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2}+\int C^{(2)}_{0,1,2}b_1^*b_2\delta_{0+1-2}\,\mathrm{d}\boldsymbol{k}_{1,2}+\int C^{(3)}_{0,1,2}b_1^*b_2^*\delta_{0+1+2}\,\mathrm{d}\boldsymbol{k}_{1,2}\\
&+\int \mathscr{N}_1 F^{(1)}_{0,1,2}\hat{\eta}_1^l b_2\delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2}+\mathrm{i}\int \mathscr{M}_1 F^{(2)}_{0,1,2}\coth[k_1h]\hat{\phi}_1^l b_2\delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2}\\
&+\int \mathscr{N}_1 F^{(3)}_{0,1,2}\hat{\eta}_1^l b_2^*\delta_{0-1+2}\,\mathrm{d}\boldsymbol{k}_{1,2}+\mathrm{i}\int \mathscr{M}_1 F^{(4)}_{0,1,2}\coth[k_1h]\hat{\phi}_1^l b_2^*\delta_{0-1+2}\,\mathrm{d}\boldsymbol{k}_{1,2}+\cdots.
\end{aligned} \tag{2.15}$$

We have omitted the details about how to treat the third-order terms in (2.14), since this involves lengthy calculations that are covered in Krasitskii (1994). Introducing

(2.15) into (2.14) and collecting the terms give

$$
\begin{aligned}
\mathrm{i}\frac{\partial b_0}{\partial t} - \omega_0 b_0 = &\int [U^{(1)}_{0,1,2} + C^{(1)}_{0,1,2}\Delta_{0-1-2}] b_1 b_2 \delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2} \\
&+ \int [2U^{(1)}_{2,1,0} + C^{(2)}_{0,1,2}\Delta_{0+1-2}] b_1^* b_2 \delta_{0+1-2}\,\mathrm{d}\boldsymbol{k}_{1,2} \\
&+ \int [U^{(3)}_{0,1,2} + C^{(3)}_{0,1,2}\Delta_{0+1+2}] b_1^* b_2^* \delta_{0+1+2}\,\mathrm{d}\boldsymbol{k}_{1,2} \\
&+ \mathrm{i}\int [4S^{(1)}_{0,1,2} + F^{(2)}_{0,1,2}\Delta_{0-2} - F^{(1)}_{0,1,2}\omega_1] \mathscr{M}_1 \coth[k_1 h] \hat{\phi}^l_1 b_2 \delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2} \\
&+ \int [4S^{(2)}_{0,1,2} + F^{(1)}_{0,1,2}\Delta_{0-2} - F^{(2)}_{0,1,2}\omega_1] \mathscr{N}_1 \hat{\eta}^l_1 b_2 \delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2} \\
&+ \mathrm{i}\int [4S^{(1)}_{0,1,2} + F^{(4)}_{0,1,2}\Delta_{0+2} - F^{(3)}_{0,1,2}\omega_1] \mathscr{M}_1 \coth[k_1 h] \hat{\phi}^l_1 b_2^* \delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2} \\
&+ \int [4S^{(3)}_{0,1,2} + F^{(3)}_{0,1,2}\Delta_{0+2} - F^{(4)}_{0,1,2}\omega_1] \mathscr{N}_1 \hat{\eta}^l_1 b_2^* \delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2}, \qquad (2.16)
\end{aligned}
$$

where $\Delta_{0-1-2} = \omega_0 - \omega_1 - \omega_2$, $\Delta_{0+1+2} = \omega_0 + \omega_1 + \omega_2$ and so on. Here, we have used the following relations valid to leading order:

$$\hat{\phi}^l_t = -g \tanh[kh]\hat{\eta}^l, \qquad \hat{\eta}_t = k\hat{\phi}^l.$$

An attempt to remove all terms on the right-hand side of (2.16) gives the following choices for $C^{(1,2,3)}_{0,1,2}$ and $F^{(1,2,3,4)}_{0,1,2}$:

$$
\begin{gathered}
C^{(1)}_{0,1,2} = -U^{(1)}_{0,1,2}\Delta^{-1}_{0-1-2}, \quad C^{(2)}_{0,1,2} = -2U^{(1)}_{2,1,0}\Delta^{-1}_{0+1-2}, \quad C^{(3)}_{0,1,2} = -U^{(3)}_{0,1,2}\Delta^{-1}_{0+1+2}, \\
F^{(1)}_{0,1,2} = -4\frac{\omega_1 S^{(1)}_{0,1,2} + \Delta_{0-2}S^{(2)}_{0,1,2}}{\Delta^2_{0-2} - \omega_1^2}, \quad F^{(2)}_{0,1,2} = -4\frac{\omega_1 S^{(2)}_{0,1,2} + \Delta_{0-2}S^{(1)}_{0,1,2}}{\Delta^2_{0-2} - \omega_1^2}, \\
F^{(3)}_{0,1,2} = -4\frac{\omega_1 S^{(1)}_{0,1,2} + \Delta_{0+2}S^{(3)}_{0,1,2}}{\Delta^2_{0+2} - \omega_1^2}, \quad F^{(4)}_{0,1,2} = -4\frac{\omega_1 S^{(3)}_{0,1,2} + \Delta_{0+2}S^{(1)}_{0,1,2}}{\Delta^2_{0+2} - \omega_1^2}.
\end{gathered}
$$

It is clear that these transformations are only valid if the denominators are never zero. It can be argued that because of the form of the gravity wave dispersion relation, the equations $\boldsymbol{k}_0 \pm \boldsymbol{k}_1 \pm \boldsymbol{k}_2 = 0$, $\omega_0 \pm \omega_1 \pm \omega_2 = 0$ have no solutions, and thus the removal of second-order terms is valid. However, because of the presence of the long wave, quadratic interactions of a long and a short wave can have periodicity close to the one of the linear short wave, and the removal of these terms becomes more suspicious. This fact is seen in the form of the functions $F^{(1)}_{0,1,2}$ and $F^{(2)}_{0,1,2}$, where the denominator has a form that can attain values very close to zero when the long-wave frequency becomes small. This requires us to keep the integrals involving $S^{(1)}_{0,1,2}$ and $S^{(2)}_{0,1,2}$ in (2.14), i.e. setting $F^{(1)}_{0,1,2} = F^{(2)}_{0,1,2} = 0$ in (2.15). Thus the resulting reduced equation is in the form

$$
\begin{aligned}
\mathrm{i}\frac{\partial b_0}{\partial t} = \omega_0 b_0 &+ \int \tilde{V}^{(2)}_{0,1,2,3} b_1^* b_2 b_3 \delta_{0+1-2-3}\,\mathrm{d}\boldsymbol{k}_{1,2,3} + 4\mathrm{i}\int \mathscr{M}_1 S^{(1)}_{0,1,2} \coth[k_1 h]\hat{\phi}^l_1 b_2 \delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2} \\
&+ 4\int \mathscr{N}_1 S^{(2)}_{0,1,2}\hat{\eta}^l_1 b_2 \delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2}. \qquad (2.17)
\end{aligned}
$$

The first part of this equation is the standard Zakharov equation, and the function $\tilde{V}^{(2)}_{0,1,2,3}$ can be found in for example Krasitskii (1994). The equation can be considered as a swell-modified version of the Zakharov equation, and (2.17) might be convenient in some applications, although it here merely acts as a step in the derivation of a swell-modified NLS equation.

2.2. *A swell-modified nonlinear Schrödinger equation*

Starting from (2.17) and assuming that the short waves are narrow banded we derive a swell-modified version of the broader bandwidth modified NLS equation (BMNLS equation) of Trulsen & Dysthe (1996).

We will mainly follow the approach of Stiassnie (1984), who showed that the Dysthe NLS equation (modified NLS equation) could be obtained as a narrow-band limit of the Zakharov equation. In the following the water depth is assumed to be large, so that the short waves can be considered deep-water waves. More specifically, we will assume $(k_s h)^{-1} = O(\epsilon)$. This, however, still allows the swell to be on intermediate depth, $(Kh)^{-1} = O(1)$.

Assuming that the energy of the short waves is concentrated around $\boldsymbol{k}_s = (k_s, 0)$ we rewrite the wave vectors as $\boldsymbol{k} = \boldsymbol{k}_s + \boldsymbol{\chi}$, where $\boldsymbol{\chi} = (\chi, \lambda)$ and $|\boldsymbol{\chi}|/k_s = O(\epsilon^{1/2})$. This scaling assumption for the bandwidth is the same as in Trulsen & Dysthe (1996). Similarly, for the swell we can write $\boldsymbol{K} = (K_x, K_y)$, where it follows from the scaling assumptions (2.2) that $K/k_s = O(\epsilon)$.

We now realize that the surface elevation of the short waves can be written in the form

$$\eta^s(\boldsymbol{x}, t) = \frac{1}{2}\left[\frac{1}{2\pi}\int \hat{B}(\boldsymbol{\chi})\mathrm{e}^{\mathrm{i}\boldsymbol{\chi}\cdot\boldsymbol{x}}\mathrm{e}^{\mathrm{i}(k_s x-\omega_s t)}\,\mathrm{d}\boldsymbol{\chi} + *\right] = \frac{1}{2}\left[B(\boldsymbol{x})\mathrm{e}^{\mathrm{i}(k_s x-\omega_s t)} + *\right],$$

where $\hat{B}(\boldsymbol{\chi}, t) = 2\mathscr{M}(\boldsymbol{k}_s + \boldsymbol{\chi})b(\boldsymbol{k}_s + \boldsymbol{\chi})\mathrm{e}^{\mathrm{i}\omega_s t}$. It is clear that B is actually a variable that the NLS equations are commonly expressed in. Further we introduce the variable $\hat{B}$ as well as the wavenumber separation indicated above into (2.17). This gives

$$\begin{aligned}\frac{\partial \hat{B}_0}{\partial t} + \mathrm{i}(\omega_0 - \omega_s)\hat{B}_0 = &-2\mathrm{i}\int \mathscr{M}_0\mathscr{N}_1\mathscr{N}_2\mathscr{N}_3\tilde{V}^{(2)}_{0,1,2,3}\hat{B}^*_1\hat{B}_2\hat{B}_3\delta_{0+1-2-3}\,\mathrm{d}\boldsymbol{\chi}_{1,2,3}\\ &+8\int \mathscr{M}_0\mathscr{M}_1\mathscr{N}_2 S^{(1)}_{0,1,2}\coth[K_1 h]\hat{\phi}^l_1\hat{B}_2\delta_{0-1-2}\,\mathrm{d}\boldsymbol{K}_1\mathrm{d}\boldsymbol{\chi}_2\\ &-8\mathrm{i}\int \mathscr{M}_0\mathscr{N}_1\mathscr{N}_2 S^{(2)}_{0,1,2}\hat{\eta}^l_1\hat{B}_2\delta_{0-1-2}\,\mathrm{d}\boldsymbol{K}_1\mathrm{d}\boldsymbol{\chi}_2.\end{aligned} \tag{2.18}$$

Taking the inverse Fourier transform of this equation with respect to $\boldsymbol{\chi}_0$ gives

$$\begin{aligned}&\frac{\partial B}{\partial t} + \mathscr{D}(B) + \frac{2i}{2\pi}\int \mathscr{M}_{3+2-1}\mathscr{N}_1\mathscr{N}_2\mathscr{N}_3\tilde{V}^{(2)}_{3+2-1,1,2,3}\hat{B}^*_1\hat{B}_2\hat{B}_3\mathrm{e}^{-\mathrm{i}\boldsymbol{\chi}_1\cdot\boldsymbol{x}}\mathrm{e}^{\mathrm{i}\boldsymbol{\chi}_2\cdot\boldsymbol{x}}\mathrm{e}^{\mathrm{i}\boldsymbol{\chi}_3\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{\chi}_{1,2,3}\\ &-\frac{1}{(2\pi)^2}\int 16\pi\mathscr{M}_{1+2}\mathscr{M}_1\mathscr{N}_2 S^{(1)}_{1+2,1,2}\coth[K_1 h]\hat{\phi}^l_1\hat{B}_2\mathrm{e}^{\mathrm{i}\boldsymbol{K}_1\cdot\boldsymbol{x}}\mathrm{e}^{\mathrm{i}\boldsymbol{\chi}_2\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{K}_1\mathrm{d}\boldsymbol{\chi}_2\\ &+\frac{1}{(2\pi)^2}\int 16\pi\mathrm{i}\mathscr{M}_{1+2}\mathscr{N}_1\mathscr{N}_2 S^{(2)}_{1+2,1,2}\hat{\eta}^l_1\hat{B}_2\mathrm{e}^{\mathrm{i}\boldsymbol{K}_1\cdot\boldsymbol{x}}\mathrm{e}^{\mathrm{i}\boldsymbol{\chi}_2\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{K}_1\mathrm{d}\boldsymbol{\chi}_2,\end{aligned} \tag{2.19}$$

where $\mathscr{D}(B)$ denotes the linear dispersive terms which are obtained by expanding $\omega(\boldsymbol{k}_s + \boldsymbol{\chi})$ in a Taylor series around $\boldsymbol{k}_s$. The first integral on the right-hand side of (2.19) is the standard term in the Zakharov equation, and the procedure of inverting this term in order to get the nonlinear terms in the modified NLS equation is basically given in Stiassnie (1984). The last two terms, however, are new and will be treated in more detail. Taylor expansion of the kernel functions gives

$$-16\pi\mathscr{M}_{1+2}\mathscr{M}_1\mathscr{N}_2 S^{(1)}_{1+2,1,2} = -k_s K_x - \chi K_x - \lambda K_y - \frac{3}{4}K_x^2 - \frac{1}{2}K_y^2 + \cdots,$$

$$16\pi\mathrm{i}\mathscr{M}_{1+2}\mathscr{N}_1\mathscr{N}_2 S^{(2)}_{1+2,1,2} = -\frac{\mathrm{i}k_s}{2\omega_s}\Omega^2 - \frac{1}{4\omega_s}\Omega^2\chi + \cdots.$$

By using the leading-order relation $\Omega^2\hat{\eta}^l = -\hat{\eta}^l_{tt}$, the above-given Taylor expansions allow us to calculate the remaining integrals in (2.19). Together with the results of Stiassnie (1984), we obtain the swell-modified BMNLS equation in the form

$$\begin{aligned}
&\frac{k_s}{\omega_s}\frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial B}{\partial x} + \frac{\mathrm{i}}{8k_s}\frac{\partial^2 B}{\partial x^2} - \frac{\mathrm{i}}{4k_s}\frac{\partial^2 B}{\partial y^2} - \frac{1}{16k_s^2}\frac{\partial^3 B}{\partial x^3} + \frac{3}{8k_s^2}\frac{\partial^3 B}{\partial x\partial y^2} - \frac{5\mathrm{i}}{128k_s^3}\frac{\partial^4 B}{\partial x^4} \\
&+ \frac{15\mathrm{i}}{32k_s^3}\frac{\partial^4 B}{\partial x^2\partial y^2} - \frac{3\mathrm{i}}{32k_s^3}\frac{\partial^4 B}{\partial y^4} + \frac{7}{256k_s^4}\frac{\partial^5 B}{\partial x^5} - \frac{35}{64k_s^4}\frac{\partial^5 B}{\partial x^3\partial y^2} + \frac{21}{64k_s^4}\frac{\partial^5 B}{\partial x\partial y^4} + \frac{\mathrm{i}k_s^3}{2}|B|^2 B \\
&+ \frac{3k_s^2}{2}|B|^2\frac{\partial B}{\partial x} + \frac{k_s^2}{4}B^2\frac{\partial B^*}{\partial x} + \frac{\mathrm{i}k_s^2}{\omega_s}B\frac{\partial\bar{\phi}}{\partial x} + \frac{\mathrm{i}k_s^2}{\omega_s}B\frac{\partial\phi^l}{\partial x} + \frac{k_s}{\omega_s}\frac{\partial B}{\partial x}\frac{\partial\phi^l}{\partial x} + \frac{k_s}{\omega_s}\frac{\partial B}{\partial y}\frac{\partial\phi^l}{\partial y} \\
&+ \frac{3k_s}{4\omega_s}B\frac{\partial^2\phi^l}{\partial x^2} + \frac{k_s}{2\omega_s}B\frac{\partial^2\phi^l}{\partial y^2} + \frac{\mathrm{i}k_s^2}{2\omega_s^2}B\frac{\partial^2\eta^l}{\partial t^2} + \frac{k_s}{4\omega_s^2}\frac{\partial B}{\partial x}\frac{\partial^2\eta^l}{\partial t^2} = 0, \quad z = 0, \qquad (2.20)
\end{aligned}$$

where $\bar{\phi}$ is governed by the equations

$$\frac{\partial^2\bar{\phi}}{\partial x^2} + \frac{\partial^2\bar{\phi}}{\partial y^2} + \frac{\partial^2\bar{\phi}}{\partial z^2} = 0, \quad z < 0, \qquad (2.21a)$$

$$\frac{k_s}{\omega_s^2}\frac{\partial^2\bar{\phi}}{\partial t^2} + \frac{\partial\bar{\phi}}{\partial z} = \frac{\omega_s}{2}\frac{\partial|B|^2}{\partial x} + \frac{\omega_s}{8k_s}\left(\mathrm{i}B\frac{\partial^2 B^*}{\partial x^2} + 2\mathrm{i}B\frac{\partial^2 B^*}{\partial y^2} + *\right), \quad z = 0, \qquad (2.21b)$$

$$\frac{\partial\bar{\phi}}{\partial z} = 0, \quad z \to -\infty. \qquad (2.21c)$$

Equations (2.20)–(2.21*c*) have previously been derived by Gramstad (2006) using a multiple-scale perturbation approach starting from the Euler equations.

In addition to the equations above, we also need an equation for the swell. Basically, the valid equations are the basic equations (2.3*a*) and (2.3*b*) applied on the swell only. However, it is clear that under the current scaling assumptions it is sufficient to consider the leading-order solution for the swell. Thus, the swell can be chosen in the form

$$\phi^l(\boldsymbol{x}, z, t) = \frac{1}{2\pi}\int F(\boldsymbol{k})\frac{\omega(\boldsymbol{k})}{\mathrm{i}k}\frac{\cosh[k(z+h)]}{\sinh[kh]}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}\,\mathrm{d}\boldsymbol{k} + *, \qquad (2.22a)$$

$$\eta^l(\boldsymbol{x}, t) = \frac{1}{2\pi}\int F(\boldsymbol{k})\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}\,\mathrm{d}\boldsymbol{k} + *, \qquad (2.22b)$$

for some suitable $F(\boldsymbol{k})$.

From the solution of the system (2.20)–(2.22*b*) the full surface elevation can be obtained as

$$\eta(\boldsymbol{x}, t) = \bar{\eta} + \frac{1}{2}\left[(B + \tilde{B})\mathrm{e}^{\mathrm{i}(k_s x - \omega_s t)} + B_2\mathrm{e}^{2\mathrm{i}(k_s x - \omega_s t)} + B_3\mathrm{e}^{3\mathrm{i}(k_s x - \omega_s t)} + \cdots + *\right] + \eta^l. \quad (2.23)$$

Here $\bar{\eta}$, B_2 and B_3 are bound wave contributions from quadratic and cubic interactions between short waves, while $\tilde{B}$ is the bound contributions arising from quadratic interactions between long and short waves. All the bound contributions can be found from transformation (2.15). Since $\bar{\eta}$, B_2 and B_3 take the same form as in the no-swell case, we here only provide details about how to obtain $\tilde{B}$ from (2.15). By assuming narrow bandwidth in the same manner as before, it is clear that

$$\tilde{B} = \frac{1}{(2\pi)^2}\int 4\pi \mathscr{M}_{1+2}\mathscr{N}_1\mathscr{N}_2 F^{(3)}_{-1-2,-1,2}\hat{\eta}^l_1\hat{B}_2 \mathrm{e}^{\mathrm{i}(\boldsymbol{K}_1+\boldsymbol{\chi}_2)\cdot \boldsymbol{x}}\,\mathrm{d}\boldsymbol{K}_1\mathrm{d}\boldsymbol{\chi}_2 - \frac{\mathrm{i}}{(2\pi)^2}\int 4\pi \mathscr{M}_{1+2}\mathscr{M}_1\mathscr{N}_2 F^{(4)}_{-1-2,-1,2}\coth\,[K_1 h]\hat{\phi}^l_1\hat{B}_2 \mathrm{e}^{\mathrm{i}(\boldsymbol{K}_1+\boldsymbol{\chi}_2)\cdot \boldsymbol{x}}\,\mathrm{d}\boldsymbol{K}_1\mathrm{d}\boldsymbol{\chi}_2,$$

and by Taylor expansion we find, valid to $O(\epsilon^3)$,

$$\tilde{B} = -\frac{k_s}{4\omega_s^2}B\frac{\partial^2\eta^l}{\partial t^2}. \tag{2.24}$$

The other bound wave contributions do not depend explicitly on the swell and can therefore be inferred from Trulsen & Dysthe (1996):

$$\bar{\eta} = -\frac{k_s}{\omega_s^2}\frac{\partial\bar{\phi}}{\partial t} - \frac{1}{16k_s}\frac{\partial^2|B|^2}{\partial x^2} - \frac{1}{8k_s}\frac{\partial^2|B|^2}{\partial y^2}, \tag{2.25a}$$

$$B_2 = \frac{k_s}{2}B^2 - \frac{\mathrm{i}}{2}B\frac{\partial B}{\partial x} + \frac{1}{2k_s}B\frac{\partial^2 B}{\partial y^2} - \frac{3}{4k_s}\left(\frac{\partial B}{\partial y}\right)^2, \tag{2.25b}$$

$$B_3 = \frac{3k_s^2}{8}B^3. \tag{2.25c}$$

The double-derivative terms in (2.21*b*) are not needed for a solution of (2.20) within its highest level of accuracy. However, the double-derivative terms in (2.21*b*) are needed in order to reconstruct the surface elevation in (2.25*a*) within its highest level of accuracy. The absence of the relevant double-derivative terms in Trulsen & Dysthe (1996) thus represents a systematic error as far as full reconstruction in their paper is concerned.

We note that (2.20) is formally valid for evolution up to time $O(\omega_s^{-1}\epsilon^{-2})$.

3. Modulation of monochromatic short wave by swell

In this section the simple situation of a monochromatic short wave modulated by a monochromatic swell is considered. For this case one can find an approximate analytical solution to the swell-modified NLS equation derived in § 2. It is shown that this solution corresponds to an extended version of the classical result of Longuet-Higgins & Stewart (1960), which predicts a local change in amplitude and phase of a short wave which is influenced by a long wave.

Assuming that the short-wave field is dominated by one single wave component, the free-wave complex amplitude B can be written as $B(\boldsymbol{x},t) = A_s + B'(\boldsymbol{x},t)$, where $A_s = a\mathrm{e}^{\mathrm{i}(\xi_s-(1/2)(k_s a)^2\omega_s t)}$ with a and ξ_s being two real constants and where $B'(\boldsymbol{x},t)$ contains the modulation of the short waves because of the long wave. It is assumed that $k_s|A_s| = O(\epsilon)$, $k_s|B'| = O(\epsilon^2)$ and that B' is varying on the same temporal and spatial scales as the swell, e.g. $|\nabla B'| = O(\epsilon^3)$. Inserted into (2.20), the equation for B' takes the form

$$\frac{\partial B'}{\partial t} + \frac{\omega_s}{2k_s}\frac{\partial B'}{\partial x} + \mathrm{i}k_s A_s\frac{\partial\phi^l}{\partial x} + \mathrm{i}k_s B'\frac{\partial\phi^l}{\partial x} + \frac{3}{4}A_s\frac{\partial^2\phi^l}{\partial x^2} + \frac{1}{2}A_s\frac{\partial^2\phi^l}{\partial y^2} + \frac{\mathrm{i}k_s}{2\omega_s}A_s\frac{\partial^2\eta^l}{\partial t^2} = 0, \quad z=0. \tag{3.1}$$

We choose the swell as a single-frequency wave component on deep water,

$$\eta^l(\boldsymbol{x}, t) = A\cos\theta_l, \quad \phi^l(\boldsymbol{x}, z, t) = \frac{A\Omega}{K}\mathrm{e}^{Kz}\sin\theta_l, \tag{3.2}$$

where $\theta_l = K_x x + K_y y - \Omega t + \xi_l$ is the phase function of the swell. By means of a perturbation expansion, an approximate solution to (3.1), valid to $O(\epsilon^3)$, is found in the form

$$B' = k_s A_s A\left[\mathrm{i}\left(\frac{K_x}{K} - \frac{1}{2}\sqrt{\frac{K}{k_s}}\left[1 - \frac{K_x^2}{K^2}\right] - \frac{1}{4}\frac{K_x}{k_s}\left[1 - \frac{K_x^2}{K^2}\right]\right)\sin\theta_l \right. \\ \left. + \frac{1}{4}\frac{K}{k_s}\left[3 - \frac{K_y^2}{K^2}\right]\cos\theta_l + Ak_s\frac{K_x^2}{4K^2}\cos(2\theta_l)\right]. \tag{3.3}$$

So far only contributions from the evolution equation itself (modifications to the free-wave complex amplitude, B) have been considered. However we also need to take the bound wave contributions into account. From (2.24)–(2.25*c*) we find

$$\tilde{B} = -\frac{k_s}{4\omega_s^2}B\frac{\partial^2\eta^l}{\partial t^2} = k_s A_s A\frac{K}{4k_s}\cos\theta_l, \qquad B_2 = \frac{k_s}{2}A_s^2, \qquad B_3 = \frac{3k_s^2}{8}A_s^3. \tag{3.4}$$

Now, using (2.23), the surface elevation is found as

$$\eta = \frac{1}{2}\left[(A_s + B' + \tilde{B})\mathrm{e}^{\mathrm{i}(k_s x - \omega_s t)} + B_2\mathrm{e}^{2\mathrm{i}(k_s x - \omega_s t)} + B_3\mathrm{e}^{3\mathrm{i}(k_s x - \omega_s t)} + *\right] + \eta^l \\ = a\cos\theta_s + \frac{k_s a^2}{2}\cos 2\theta_s + \frac{3k_s^2 a^3}{8}\cos 3\theta_s + A\cos\theta_l \\ - aAk_s\left(\frac{K_x}{K} - \frac{1}{2}\sqrt{\frac{K}{k_s}}\left[1 - \frac{K_x^2}{K^2}\right] - \frac{1}{4}\frac{K_x}{k_s}\left[1 - \frac{K_x^2}{K^2}\right]\right)\sin\theta_s\sin\theta_l \\ + aAK\left(1 - \frac{1}{4}\frac{K_y^2}{K^2}\right)\cos\theta_s\cos\theta_l + a(Ak_s)^2\frac{K_x^2}{4K^2}\cos\theta_s\cos(2\theta_l), \tag{3.5}$$

where $\theta_s = k_s x - \omega_s t - (1/2)(k_s a)^2\omega_s t + \xi_s$ is the phase function of the main short wave. If we now rewrite (3.5) into the form

$$\eta(\boldsymbol{x}, t) = a'\cos(\theta_s + \chi) + \frac{k_s a^2}{2}\cos 2\theta_s + \frac{3k_s^2 a^3}{8}\cos 3\theta_s + A\cos\theta_l, \tag{3.6}$$

it is found that the modified amplitude of the short wave is

$$a' = a\left[1 + AK\left(1 - \frac{1}{4}\frac{K_y^2}{K^2}\right)\cos\theta_l + (Ak_s)^2\frac{K_x^2}{4K^2}\right]. \tag{3.7}$$

The second term in (3.7) is an extension to two horizontal dimensions of the result of Longuet-Higgins & Stewart (1960) and predicts that the short-wave amplitude is increased close to the crests of the long wave and decreased close to the troughs of the long wave. However, because of differences in the scaling assumptions between the present approach and that of Longuet-Higgins & Stewart (1960), we also obtain an additional term that predicts a constant increase in short-wave amplitude independent of the long-wave phase. This term does not appear in Longuet-Higgins & Stewart (1960).

4. Statistics for linear random short waves modulated by a swell

In the previous section an approximate analytical solution which describes the modification of a short wave due to a swell was found. If we ignore the mutual interaction between different short-wave components, this result can be generalized to the case of a random field of short waves influenced by a swell. Thus, we assume that a given sea state consists of one system of random short waves and one monochromatic swell. In the case of no swell it is well known that the sea surface is Gaussian distributed with Rayleigh-distributed amplitude and uniformly distributed phase. Because of the swell, however, these distributions may change. In the following we will consider the distributions and statistical properties of the swell-modified sea.

We first address the distribution of the amplitude. On the basis of the amplitude modification (3.7) found in the previous section, we write $a' = a(1 + c\cos\theta_l + d)$, where $c = AK(1 - K_y^2/4K^2)$ and $d = (Ak_s)^2 K_x^2/4K^2$. We note that both c and d are $O(\epsilon^2)$. Assuming that a is Rayleigh distributed with parameter σ and that θ_l is uniformly distributed on the interval $[0, 2\pi)$, one can find the probability density function (p.d.f.) of a' in the form

$$f_{a'}(z) = \int_{1+d-c}^{1+d+c} \frac{z}{(\sigma y)^2} \exp\left[-\frac{z^2}{2(\sigma y)^2}\right] \frac{1}{\pi\sqrt{c^2 - (y-1-d)^2}}\,\mathrm{d}y. \tag{4.1}$$

We note that (4.1) can be written as

$$f_{a'}(z) = \int_{1+d-c}^{1+d+c} \frac{g(z, y)}{\pi\sqrt{c^2 - (y-1-d)^2}}\,\mathrm{d}y, \tag{4.2}$$

where $g(z, y)$ is the Rayleigh distribution with parameter σy. Representation (4.2) can be used to express the p.d.f. as an asymptotic expansion in the small parameter c:

$$f_{a'}(z) = \sum_{n=0}^{\infty} \left(\frac{c}{2}\right)^{2n} \frac{g^{(2n)}(z, 1+d)}{(n!)^2}, \tag{4.3}$$

where $g^{(2n)} = \partial^{2n} g/\partial y^{2n}$. Since $c = O(\epsilon^2)$, it is consistent to only consider the leading-order term in (4.3). Thus, the swell-modified amplitude distribution takes the form

$$f_{a'}(z) = \frac{z}{(\sigma[1+d])^2} \exp\left[-\frac{z^2}{2(\sigma[1+d])^2}\right] + O(c^2). \tag{4.4}$$

In a similar manner, we can use the expression for the surface elevation (3.5), to find the statistical properties of the surface elevation. Excluding the terms corresponding to bound modes of the short wave only, we write (3.5) as

$$\eta = a[\cos\theta_s - b\sin\theta_l \sin\theta_s + c\cos\theta_l\cos\theta_s + d\cos 2\theta_l \cos\theta_s].$$

Here, a, θ_s and θ_l are assumed to be independent random variables; a is Rayleigh distributed, while θ_s and θ_l are uniformly distributed on the interval $[0, 2\pi)$. Using that $X = a\cos\theta_s$ and $Y = a\sin\theta_s$ are independent and normally distributed with mean 0 and variance σ^2, one can easily find that the first four moments of η take the forms

$$E[\eta] = E[\eta^3] = 0, \quad E[\eta^2] = \sigma^2\left(1 + \frac{b^2}{2}\right), \quad E[\eta^4] = 3\sigma^4(1 + b^2). \tag{4.5}$$

The expressions for the moments (4.5) are valid to $O(\epsilon^2)$. Using (4.5) it is actually found that the kurtosis become $\kappa = E[\eta^4]/E[\eta^2]^2 = 3 + O(b^4)$, which means that to the present order of accuracy there is no change in the kurtosis due to the swell.

Finally we find how the probability of freak waves is changed by the swell. A common definition of a freak wave is a wave whose crest height exceeds 1.25 times the significant wave height of the given sea state (Dysthe, Krogstad & Müller 2008). Thus the freak wave probability can be defined as $\Pr(a > 5\sigma)$. Using (4.4) and (4.5), one obtains

$$\Pr(a' > 5\sigma\sqrt{1+b^2/2}) = \int_{5\sigma\sqrt{1+b^2/2}}^{\infty} \frac{z}{(\sigma[1+d])^2} \exp\left[-\frac{z^2}{2(\sigma[1+d])^2}\right] \mathrm{d}z$$
$$= \mathrm{e}^{-25/2}\left(1 + 25d - \frac{25}{4}b^2\right). \tag{4.6}$$

Thus, at first glance it appears that the swell changes the freak wave probability slightly. However, when inserting the values for b and d, some cancellations occur, and it is found that $25d - 25b^2/4 = O(\epsilon^3)$. That is to say like for the kurtosis there is no change in the freak wave probability to the order of accuracy considered here. Thus, the simplified analysis in this section indicates that the effect of the swell on the occurrence of extreme waves is a very small effect.

5. Numerical simulations

The findings of §§ 3 and 4 indicate that the swell has only minor impact on the statistical distribution of short waves. However §§ 3 and 4 are based on simplified conditions which do not include all effects described by the swell-modified equation derived in § 2. In this section numerical simulations with the full equation are performed, from which statistical information is extracted. The numerical simulations show a somewhat stronger effect of the swell than what is indicated by the simplified analysis in §§ 3 and 4. Furthermore, in contrast to §§ 3 and 4, the numerical simulations show that the swell effect depends on the shape of the short-wave spectrum. This indicates that nonlinear effects are important for the evolution of short waves influenced by a swell.

5.1. *Numerical method*

To solve the swell-modified NLS equation numerically in a rectangular periodic domain a Runge–Kutta scheme with variable time step (Matlab's ode45 solver) is employed for the integration in time. Because of the rectangular domain with periodic boundary conditions all the spatial derivatives can be evaluated by using fast Fourier transform routines, which makes the numerical solver quite efficient.

An alternative implementation based on the split-step method commonly used for solving NLS-type equations (see e.g. Lo & Mei 1985, 1987; Weideman & Herbst 1986) was also tested. This alternative implementation produced identical results as the Runge–Kutta scheme. However here the Runge–Kutta scheme turned out to be more efficient for obtaining the same accuracy of the solution. The accuracy of the numerical solution is controlled internally by the solver, in agreement with pre-chosen relative and absolute error tolerances. The solver chooses the time step so that the error in the solution at every time step is smaller than the error tolerances. Here we chose the relative and absolute error tolerances to be 10^{-3} and 10^{-6} respectively. This choice led to conservation of the wave action within 0.1 % of its initial value.

A computational domain of $n_x \times n_y = 128 \times 128$ points, which corresponds to about 64×64 characteristic short wavelengths, has been used to represent the free-wave complex amplitude B. When reconstructing the surface elevation, including bound modes, we have used a discretization of 768×384 points to represent the fully reconstructed surface.

In the time domain we integrate the equation up to $t = 100T_p$, which is consistent with the time horizon for which the equations are formally valid. This allows us to describe the relatively rapid nonlinear interactions happening on the Benjamin–Feir time scale. Processes taking place over larger times are however not considered here.

5.2. *Initial condition*

As initial condition, the free-wave complex amplitude is chosen according to a spectrum of independent Fourier modes with random phases and amplitudes. The amplitudes are chosen in order to give the desired wave spectrum. For all the results presented here a JONSWAP wavenumber spectrum with a directional distribution $D(\xi)$ has been employed. Thus in the wave vector plane $(k_x, k_y) = (k\cos\xi, k\sin\xi)$ the spectrum has the form

$$S(k_x, k_y) = \frac{\alpha}{k^3}\exp\left[-\frac{5}{4}(k/k_s)^{-2}\right]\gamma^{\exp[-(\sqrt{k/k_s}-1)^2/(2\sigma_A^2)]}\frac{D(\xi)}{k}, \tag{5.1}$$

where the directional spreading function $D(\xi)$ has the form

$$D(\xi) = \frac{\Gamma\left(\dfrac{N}{2}+1\right)}{\sqrt{\pi}\Gamma\left(\dfrac{N}{2}+\frac{1}{2}\right)}\cos^N\xi. \tag{5.2}$$

The parameter σ_A is equal to 0.07 for $k \leqslant k_s$ and 0.09 for $k > k_s$, and the parameters α, γ and N were varied to give the desired spectral shape and wave steepness. In the numerical simulations the spectra were chosen so that $k_s\sqrt{\overline{|B|^2}} = 0.1$; i.e. the steepness of the first-order reconstructed surface is set to 0.1. When accounting for bound waves, the fully reconstructed surface thus has a slightly larger steepness.

It is well established that the statistical properties of a nonlinear wave field are strongly related to the shape of the underlying wave spectrum. In particular, it has been suggested that the two most important parameters for predicting large deviation from linear statistics in a nonlinear wave field are the BFI, i.e. the ratio of wave steepness to spectral bandwidth, and the degree of directional spreading, i.e. the crest length, of the waves. In the present work the main objective is to investigate how the presence of a swell might change the statistics of nonlinear short waves. Since the statistics depends so strongly on the characteristics of the wave spectrum, a more complete picture is given if the effect of the swell is investigated for several different types of wave spectra for the short waves. In the following, results from four different types of initial spectra will be presented, and the parameters γ and N are chosen so that the four cases include both small and large values of both the BFI and the crest length. In this way the four most important regions in the BFI/crest-length plane are considered (see e.g. Gramstad & Trulsen 2007; Onorato *et al.* 2009*a*,*b*). We anticipate that these four cases are sufficient in order to characterize the dependence on the spectral shape of the short waves.

The values of the different parameters in each case are given in table 1. In table 1 the BFI is defined in terms of the frequency bandwidth, $\text{BFI} = \epsilon\omega_s/\Delta\omega$, where $\Delta\omega$ is calculated as the half-width at half-maximum of the frequency spectrum $S_\omega(\omega)$.

Case	γ	N	BFI	L_c
A	2.5	10	0.90	2.84
B	6	10	1.34	2.81
C	2.5	840	0.90	27.8
D	6	840	1.34	25.0

TABLE 1. Parameters of the JONSWAP spectra for the four different scenarios.

The crest length L_c is defined as $L_c = k_s/\Delta k_y$, where Δk_y is the half-width at half-maximum in the k_y direction of the wave vector spectrum $S(k_x, k_y)$ (see, e.g., Gramstad & Trulsen 2007).

Cases A and B correspond to waves with a relatively broad directional spreading. Such a broad directional spreading is shown to give statistical properties close to what is expected according to linear (Gaussian) theory (Gramstad & Trulsen 2007). Moreover it is expected that the results are only weakly affected by the BFI.

For cases C and D a much narrower directional spreading is applied. The choice of $N = 840$ corresponds to a crest length of about 25 characteristic wavelengths. It was suggested by Gramstad & Trulsen (2007) that there exists a qualitative transition between long-crested waves (with large deviations from Gaussian statistics) and short-crested waves (with small deviation from Gaussian statistics) for a crest length of about 10 characteristic wavelengths. Cases C and D are thus well above the limit of $L_c = 10$, and consequently larger deviations from Gaussian statistics are expected.

The swell can in principle be chosen as any solution in the form (2.22*a*)–(2.22*b*) that is in agreement with the relevant scaling assumptions (2.2). Here, however, a simple swell in the form of a single plane wave is employed, i.e. $F(\boldsymbol{k}) = \pi A\delta(\boldsymbol{k} - \boldsymbol{K})$ in (2.22*a*) and (2.22*b*), which gives

$$\eta^l(\boldsymbol{x}, t) = A\cos(K_x x + K_y y - \Omega t), \tag{5.3a}$$

$$\phi^l(\boldsymbol{x}, z, t) = \frac{A\Omega}{K}\frac{\cosh[K(z+h)]}{\sinh[Kh]}\sin(K_x x + K_y y - \Omega t). \tag{5.3b}$$

The choice of A, K and $\Omega = \omega(\boldsymbol{K})$ is restricted by the underlying scaling assumptions. The values $A/a = 1$ and $K/k_s = 0.3$ have been chosen, which correspond to a swell with steepness $AK = 0.03$. Five different directions of swell propagation relative to the short waves were considered: $\theta = 0°, 45°, 90°, 135°, 180°$, where $\theta = \arccos(\boldsymbol{k}_s \cdot \boldsymbol{K}/k_s K)$ is the angle between the direction of swell propagation and the main direction of propagation for the short waves. In addition simulations without the swell are performed, for comparison with the various swell cases.

5.3. *Results*

Using the numerical solver, a Monte Carlo approach has been used to investigate how the swell affects the statistical properties of the short waves. A large number of simulations (with different random phases and amplitudes) have been performed for all the different swell directions as well as for the case without swell; 5000 simulations have been performed for each of the 24 swell/short-wave combinations (a total of 120 000 runs). This turned out to be sufficient in order to have a satisfactory convergence of the results presented in the following. The convergence is discussed briefly below.

Since the numerical solution of (2.20) provides the free-wave complex amplitude B, a subsequent reconstruction is needed in order to obtain the surface elevation η^s. The contributions from bound waves needed in this reconstruction are given in

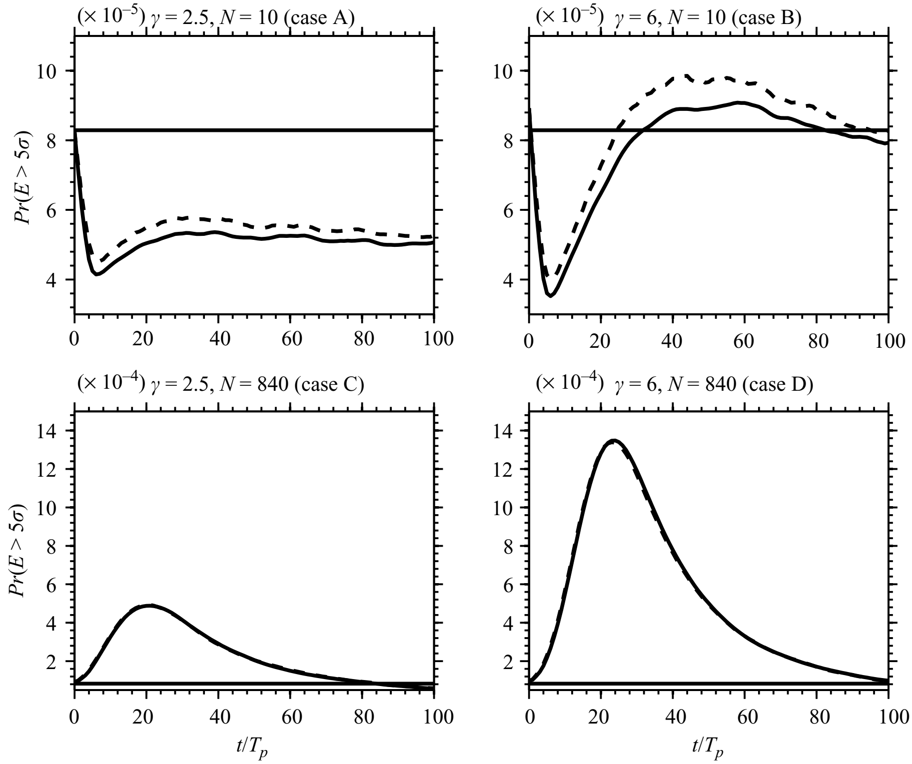

FIGURE 1. Evolution of the freak wave probability for cases A–D, based on ensemble averaging 5000 realizations for each case. Simulations without the swell (solid line); the swell and short waves propagating in the same direction, $\theta = 0$ (dashed line). The horizontal line corresponds to the Tayfun distribution.

(2.24)–(2.25*c*). All the following numerical results are obtained from the reconstructed surface, including bound modes up to third order in the short-wave steepness. However, when solving the induced mean flow problem (2.21), only the leading-order terms in (2.21*b*) are accounted for. Under the slightly broader bandwidth assumption, this formally gives a reconstruction of the zeroth harmonic bound wave consistent only to $O(\epsilon^{2.5})$.

From the 5000 Monte Carlo runs for each swell direction and for each of the cases, A–D, the statistics of the wave fields were recorded. Since the main goal here is to see whether the presence of the swell modifies the wave statistics, the simulations without the swell are used as a reference, and the simulations with the swell are compared with the no-swell case.

As a measure of the intensity of large waves, we use the relative number of crests that exceed five times the standard deviation of the surface, i.e. $\Pr(E > 5\sigma)$, where E is the upper envelope defined as $E = \bar{\eta} + |B + \tilde{B}| + |B_2| + |B_3|$ and where $\sigma = \sqrt{\overline{(\eta^s - \overline{\eta^s})^2}}$ is the standard deviation of the surface. The probability $\Pr(E > 5\sigma)$ corresponds to the common definition of a freak wave as a wave whose crest height exceeds 1.25 times the significant wave height (Dysthe *et al.* 2008), and the probability $\Pr(E > 5\sigma)$ is in the following referred to as the freak wave probability.

Figure 1 shows how the freak wave probability evolves in time for the four cases, based on ensemble averaging 5000 realizations for each case. The solid lines correspond to the no-swell case, while the dashed lines correspond to the case $\theta = 0$

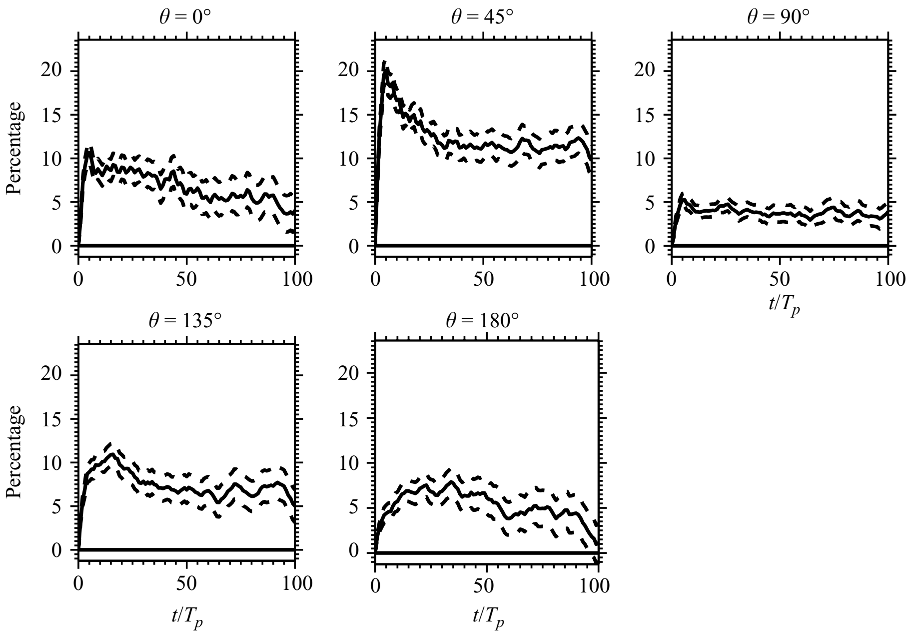

FIGURE 2. The mean relative difference in freak wave probability between the cases with and without the swell for case A, based on ensemble average over 5000 realizations. The dashed lines indicate estimated 95 % confidence intervals for this mean difference.

in which the swell and short waves propagate in the same direction. The horizontal lines correspond to the Tayfun distribution (narrow-band second-order theory).

In the upper row the two cases with a relatively broad directional distribution, cases A and B, are shown. As expected for such short-crested seas, only small deviations from the second-order theory are seen. A small effect of the BFI is seen, as case B generally has a somewhat larger freak wave probability than case A. In general, the same behaviour is seen both in the no-swell case and in the case with the swell; however, a slightly larger value of the freak wave probability is seen for the swell-case. The increase can be estimated to be in the neighbourhood of 10 %; however, our results indicate that the exact percentage depends on the exact freak wave definition used (here $E > 5\sigma$).

For the longer-crested cases shown in the lower row in figure 1 a somewhat different behaviour is seen. Here, the freak wave probability increases until a maximum is reached after about $20T_p$. This maximum is several times larger than what is predicted by second-order theory. However, after the maximum is reached the freak wave probability is decreasing and approaches values close to second-order theory during the end of the evolution. This temporary increase in freak wave probability is related to the rapid transition of the narrow initial spectrum into a broader spectrum that takes place on the Benjamin–Feir time scale. During this transition more freak waves are produced, before the number of freak waves decreases and stabilizes when the spectrum has evolved into a broader spectrum. A clear effect of the BFI is also seen in this case, as the case with largest BFI (case D) reaches a maximum more than twice the magnitude of case C. In these cases it is clear that the swell hardly modifies the freak wave probability at all, and the exact effect of the swell may even be to lower the freak wave probability by an insignificant amount.

In figures 2–5 the effect of the swell is shown in more detail. The solid lines show the mean relative difference in freak wave probability (in per cent) between

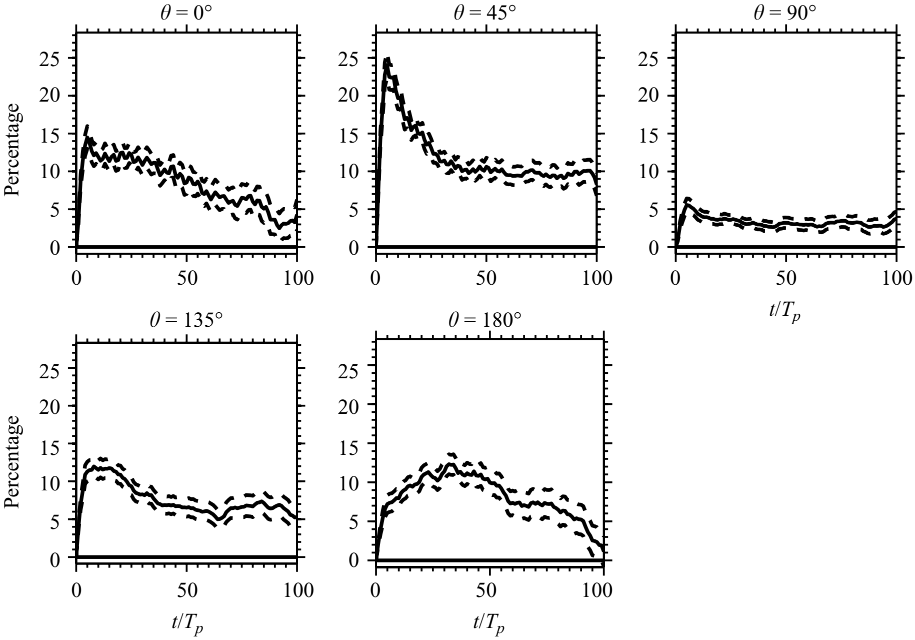

FIGURE 3. Same as figure 2 for case B.

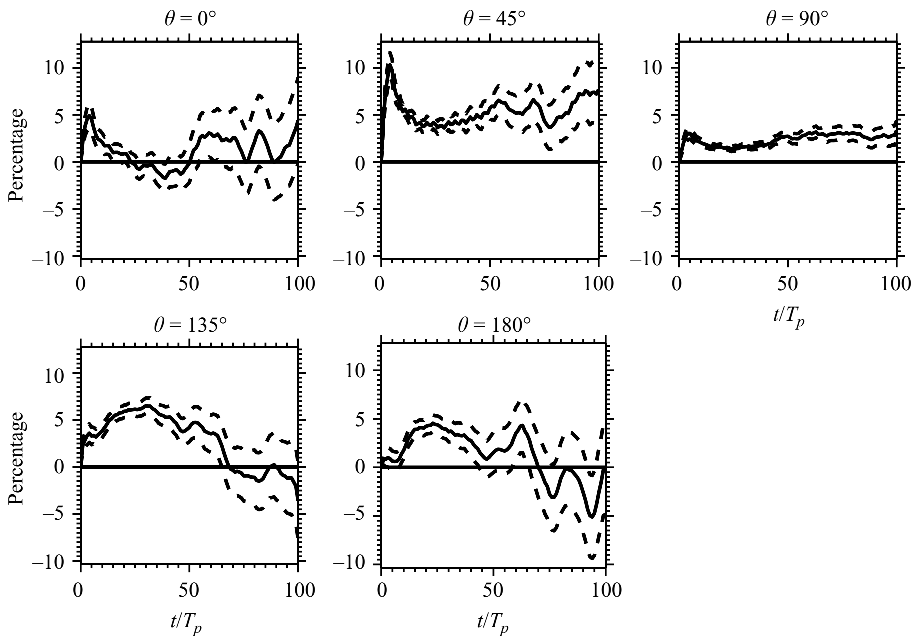

FIGURE 4. Same as figure 2 for case C.

the various swell cases and the no-swell case. The broken lines indicate approximate 95 % confidence intervals for the mean difference based on the 5000 realizations in each case. We have verified that the number of realizations employed is sufficient in order to obtain statistically stable results. In figure 6 an example of the convergence with respect to ensemble size is shown for case D with $\theta = 0^\circ$ at $t = 100T_p$. The

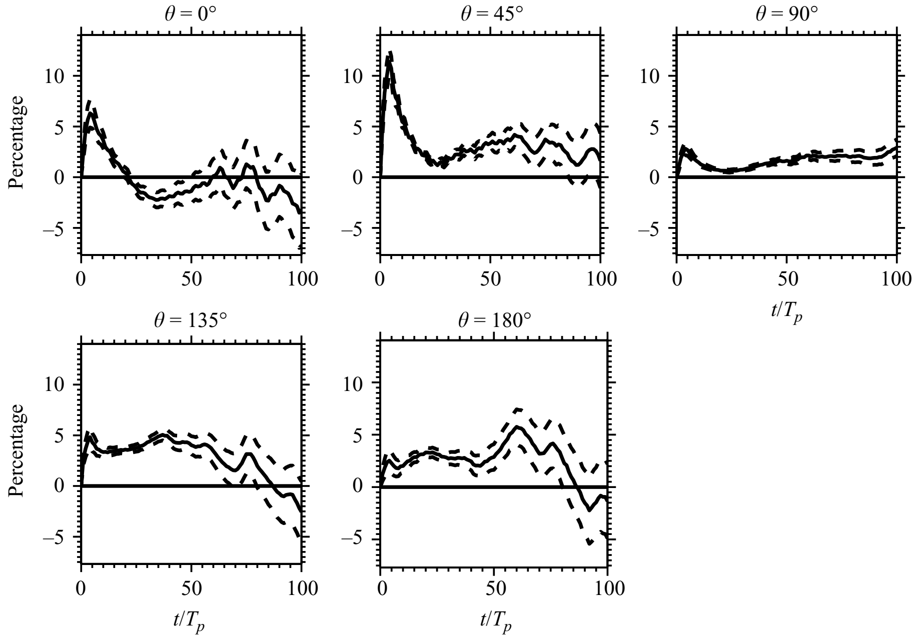

FIGURE 5. Same as figure 2 for case D.

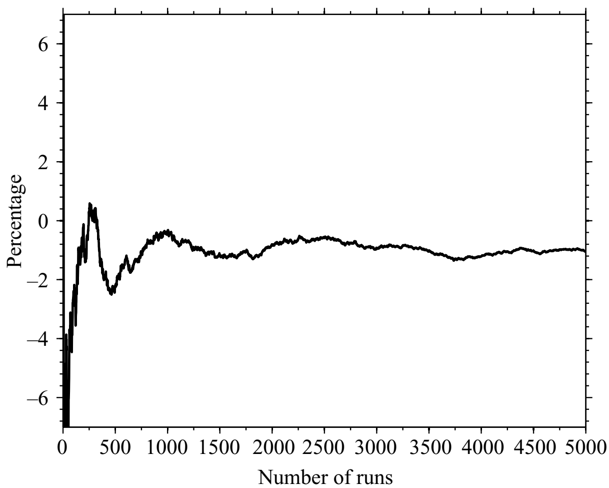

FIGURE 6. Mean relative difference in freak wave probability for case D, $\theta = 0°$ at $t = 100T_p$ as a function of the number of random runs used to calculate the mean.

figure shows a somewhat slow convergence; however using 5000 runs seems to be sufficient.

For the two short-crested cases (A and B) in figures 2 and 3 we note an increase in freak wave occurrence when a swell is present. Our results suggest that the increase is about 5–20 % and definitely less than 30 %. We have found that the exact values of these percentages depend on the freak wave definition used; we show results only for the definition $E > 5\sigma$.

For the two long-crested cases (C and D) in figures 4 and 5 the effect of the swell is less pronounced. The confidence intervals for the mean include zero during parts of the evolution; thus the average swell effect might be close to insignificant in these cases.

Taking into account that the two long-crested cases (C and D) most probably have unrealistically long crest lengths, our results therefore suggest that a swell is likely to have the potential of increasing the freak wave occurrence in more realistic seas (cases A and B).

The modification of freak wave occurrence due to swell does not seem to be sensitive to the value of the BFI, regardless of the crest length.

The most striking result seen in figures 2–5 is the dependence on the direction of swell propagation. There is clear evidence that the confidence interval is narrowest, and the change in mean freak wave occurrence is minimum, when the swell is orthogonal ($\theta = 90^\circ$) to the wind sea. Thus the speculation made by Lechuga (2006) that the swell and wind sea at right angles should not provoke enhanced freak wave occurrence in the wind sea indeed seems to hold, although we have arrived at this conclusion through a completely different path. This does however not imply that wind waves are not modified by an orthogonal swell; in fact, our extended version of the result of Longuet-Higgins & Stewart (1960) does predict modulation of the short waves in the orthogonal case, and as suggested by Henyey *et al.* (1988) and Naciri & Mei (1994), a wind sea will in any case feel the apparent gravity of a swell regardless of its direction.

6. Conclusions

By use of the swell-modified NLS equation derived in § 2 the effect of a swell on irregular and directional short waves has been investigated both through simplified analytical solution of the equations and through numerical Monte Carlo simulations of the full equations.

The simplified, linearized analysis in § 4 arrives at the conclusion that a swell hardly changes the statistical properties of the short waves at all.

In § 5 results from the Monte Carlo simulations have been presented and can be summarized as follows.

For short-crested seas we anticipate that a swell can increase the probability of freak waves by 5–20 % compared with a corresponding sea without swell. For long-crested seas we anticipate that a swell has a much smaller effect on the probability of freak waves and not necessarily that of increasing the probability of freak waves.

For orthogonal swell and short waves the modification of the number of freak waves is much less than for non-orthogonal swell and short waves.

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REFERENCES

Akhmediev, N., Ankiewicz, A. & Taki, M. 2009 Waves that appear from nowhere and disappear without a trace. *Phys. Lett.* A **373** (6), 675–678.

Alber, I. E. 1978 The effects of randomness on the stability of two-dimensional surface wavetrains. *Proc. R. Soc. Lond.* A **363** (1715), 525–546.

Craik, A. D. D. 1988 Interaction of a short-wave field with a dominant long-wave in deep-water – derivation from Zakharov's spectral formulation. *J. Austral. Math. Soc.* B **29**, 430–439.

Dysthe, K., Krogstad, H. E. & Müller, P. 2008 Oceanic rogue waves. *Annu. Rev. Fluid Mech.* **40** (1), 287–310.

Dysthe, K. B. & Trulsen, K. 1999 Note on breather type solutions of the NLS as models for freak-waves. *Phys. Scripta* **T82**, 48–52.

Fuhrman, D. R., Madsen, P. A. & Bingham, H. B. 2006 Numerical simulation of lowest-order short-crested wave instabilities. *J. Fluid Mech.* **563**, 415–441.

Gramstad, O. 2006 Kan dønning framprovosere ekstreme bølger i vindsjø? (in Norwegian) Master's thesis, Mechanic Division, Department of Mathematics, University of Oslo, Oslo, Norway.

Gramstad, O. & Trulsen, K. 2007 Influence of crest and group length on the occurrence of freak waves. *J. Fluid Mech.* **582**, 463–472.

Grimshaw, R. 1988 The modulation of short gravity-waves by long waves or currents. *J. Austral. Math. Soc.* B **29**, 410–429.

Henyey, F. S., Creamer, D. B., Dysthe, K. B., Schult, R. L. & Wright, J. A. 1988 The energy and action of small waves riding on large waves. *J. Fluid Mech.* **189**, 443–462.

Janssen, P. A. E. M. 2003 Nonlinear four-wave interactions and freak waves. *J. Phys. Oceanogr.* **33** (4), 863–884.

Krasitskii, V. P. 1994 On reduced equations in the Hamiltonian theory of weakly nonlinear surface-waves. *J. Fluid Mech.* **272**, 1–20.

Lechuga, A. 2006 Were freak waves involved in the sinking of the tanker 'Prestige'? *Nat. Haz. Earth Sys.* **6** (6), 973–978.

Lo, E. & Mei, C. C. 1985 A numerical study of water-wave modulation based on a higher-order nonlinear Schrödinger-equation. *J. Fluid Mech.* **150**, 395–416.

Lo, E. Y. & Mei, C. C. 1987 Slow evolution of nonlinear deep-water waves in two horizontal directions – a numerical study. *Wave Mot.* **9** (3), 245–259.

Longuet-Higgins, M. S. 1987 The propagation of short surface-waves on longer gravity-waves. *J. Fluid Mech.* **177**, 293–306.

Longuet-Higgins, M. S. & Stewart, R. W. 1960 Changes in the form of short gravity waves on long waves and tidal currents. *J. Fluid Mech.* **8** (4), 565–583.

Masson, D. 1993 On the nonlinear coupling between swell and wind waves. *J. Phys. Oceanogr.* **23** (6), 1249–1258.

Naciri, M. & Mei, C. C. 1992 Evolution of a short surface-wave on a very long surface-wave of finite-amplitude. *J. Fluid Mech.* **235**, 415–452.

Naciri, M. & Mei, C. C. 1993 Evolution of short gravity-waves on long gravity-waves. *Phys. Fluids* A **5** (8), 1869–1878.

Naciri, M. & Mei, C. C. 1994 Two-dimensional modulation and instability of a short wave riding on a finite-amplitude long wave. *Wave Mot.* **20** (3), 211–232.

Onorato, M., Cavaleri, L., Fouques, S., Gramstad, O., Janssen, P. A. E. M., Monbaliu, J., Osborne, A. R., Pakozdi, C., Serio, M., Stansberg, C. T., Toffoli, A. & Trulsen, K. 2009*a* Statistical properties of mechanically generated surface gravity waves: a laboratory experiment in a three-dimensional wave basin. *J. Fluid Mech.* **627**, 235–257.

Onorato, M., Osborne, A. R. & Serio, M. 2002 Extreme wave events in directional, random oceanic sea states. *Phys. Fluids* **14** (4), L25–L28.

Onorato, M., Osborne, A. R. & Serio, M. 2006 Modulational instability in crossing sea states: a possible mechanism for the formation of freak waves. *Phys. Rev. Lett.* **96** (1), 014503.

Onorato, M., Osborne, A. R., Serio, M. & Bertone, S. 2001 Freak waves in random oceanic sea states. *Phys. Rev. Lett.* **86** (25), 5831–5834.

Onorato, M., Osborne, A. R., Serio, M., Cavaleri, L., Brandini, C. & Stansberg, C. T. 2004 Observation of strongly non-Gaussian statistics for random sea surface gravity waves in wave flume experiments. *Phys. Rev.* E **70** (6), 067302.

Onorato, M., Waseda, T., Toffoli, A., Cavaleri, L., Gramstad, O., Janssen, P. A. E. M., Kinoshita, T., Monbaliu, J., Mori, N., Osborne, A. R., Serio, M., Stansberg, C. T., Tamura, H. & Trulsen, K. 2009*b* Statistical properties of directional ocean waves: the role of the modulational instability in the formation of extreme events. *Phys. Rev. Lett.* **102** (11), 114502.

Regev, A., Agnon, Y., Stiassnie, M. & Gramstad, O. 2008 Sea–swell interaction as a mechanism for the generation of freak waves. *Phys. Fluids* **20** (11), 112102.

Shukla, P. K., Kourakis, I., Eliasson, B., Marklund, M. & Stenflo, L. 2006 Instability and evolution of nonlinearly interacting water waves. *Phys. Rev. Lett.* **97** (9), 094501.

Socquet-Juglard, H., Dysthe, K., Trulsen, K., Krogstad, H. E. & Liu, J. D. 2005 Probability distributions of surface gravity waves during spectral changes. *J. Fluid Mech.* **542**, 195–216.

Stansberg, C. T. 1994 Effects from directionality and spectral bandwidth on nonlinear spatial modulations of deep-water surface gravity wave trains. In *24th International Conference on Coastal Engineering*, Kobe, Japan.

Stiassnie, M. 1984 Note on the modified nonlinear Schrödinger-equation for deep-water waves. *Wave Mot.* **6** (4), 431–433.

Tamura, H., Waseda, T. & Miyazawa, Y. 2009 Freakish sea state and swell–windsea coupling: numerical study of the Suwa-Maru incident. *Geophys. Res. Lett.* **36**, L01607.

Toffoli, A., Lefevre, J. M., Bitner-Gregersen, E. & Monbaliu, J. 2005 Towards the identification of warning criteria: analysis of a ship accident database. *Appl. Ocean Res.* **27** (6), 281–291.

Trulsen, K. & Dysthe, K. B. 1996 A modified nonlinear Schrödinger equation for broader bandwidth gravity waves on deep water. *Wave Mot.* **24** (3), 281–289.

Waseda, T. 2006 Impact of directionality on the extreme wave occurrence in a discrete random wave system. In *Ninth International Workshop on Wave Hindcasting and Forecasting*, Victoria, BC, Canada.

Weideman, J. A. C. & Herbst, B. M. 1986 Split-step methods for the solution of the nonlinear Schrödinger-equation. *SIAM J. Numer. Anal.* **23**, 485–507.

Zakharov, V. E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys.* **9**, 190–194.

Zhang, J. & Melville, W. K. 1990 Evolution of weakly nonlinear short waves riding on long gravity-waves. *J. Fluid Mech.* **214**, 321–346.

Zhang, J. & Melville, W. K. 1992 On the stability of weakly nonlinear short waves on finite-amplitude long gravity-waves. *J. Fluid Mech.* **243**, 51–72.